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PARAMETRIC ANALYSIS OF VELOCITY DISTRIBUTION OF TWO IMMISCIBLE INCOMPRESSIBLE FLUID

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ABSTRACT

In this study, a parametric analysis was carried out to determine the velocity distribution of two immiscible incompressible fluids between two stationary parallel plates using the Finite Element Method (FEM). The overall results from these finite elements were assembled to represent the velocity distribution in the entire domain of the stationary parallel plates. The results obtained showed that the velocity distribution has a parabolic profile with the maximum velocity of 13.125 m/s at 2.5m from the centre line into the less dense and less viscous region of the parallel plates. The fluid due to the no slip boundary condition had a velocity of 0 m/s at the walls of the parallel plates. The result obtained from the FEM when compared with the result obtained from the exact differential equation method showed a strong agreement.

1. INTRODUCTION

Champati and Ramana Rao, (2013) reveals that the velocity profile due to the flow of two incompressible immiscible liquids occupying equal heights between two parallel plates was first obtained by Bird et al., (1960). This problem was further generalized by Kapur and Shukla (1964) to the case of flow of a number of incompressible immiscible liquids occupying different heights. The stability analysis of two superposed fluids between parallel planes was formulated by Yih (1965) and later extended by Nakaya and Hasegawa (1974) to include the effects of gravity and surface tension. Santowski et al., (1969) studied the stability analysis by considering the stratified gas over a liquid under the assumption of inviscid and incompressible flow. This assumption was only an approximation which was true only when the fluid velocities are low. Rudraiah and Rohini (1975) modified the work of Santowski et al., (1969) by considering the superposed flow of a compressible fluid over an incompressible fluid but under the assumption that the compressibility of fluid is taken as an isothermal atmosphere where the density changes with height. Ramana Rao and Narayana (1981) studied the flow of two incompressible immiscible liquids occupying equal heights between two parallel plates in a rotating system under the action of constant pressure gradient. They also studied the associated thermal distribution, assuming equal and different plate temperatures. By immiscible fluids, we mean, superposed fluids of different densities and viscosities. Ramana Rao and Narayana (1981) suggested that olive-oil and water can be taken as the two immiscible liquids.

The uniqueness for two immiscible fluids in a one-dimensional porous medium was studied by Baiocchi et al. (1980). Champati and Ramana Rao (2013) studied the steady laminar flow of two incompressible immiscible liquids under the action of a constant pressure gradient through a channel of circular cross - section, rotating with a uniform angular velocity about an axis perpendicular to the channel in saturated porous medium based on Brinkman's Model (Brinkman, 1947). Akpobi and Akpobi (2007) used the finite element method to solve a problem on the velocity distribution in viscous incompressible fluid using the langrange interpolation function and compared their result with the exact differential equation method. Also, Erhunmwun and Oladeinde (2016) used the Finite Element Method (FEM) to determine the velocity distribution in a concentric cylindrical annulus.

In this study, FEM was used to determine the velocity distribution of two incompressible immiscible fluids with different densities and viscosities. The advantage of the FEM over other numerical methods is that FEM gives results that represent the velocities at different nodes for the whole material under consideration at the same time unlike the result from other numerical methods that provide discrete result at a time and needs further iteration to determine the velocity values at other points of the two stationary parallel plates.

2. METHODOLOGY

Consider the steady laminar flow of two immiscible incompressible fluids in a region between two parallel stationary plates under the influence of a pressure gradient. The fluid rates are adjusted such that the lower half of the region is filled with Fluid I (the denser and more viscous fluid) and the upper half is filled with Fluid II (the less dense and less viscous fluid), as shown in Figure 1.

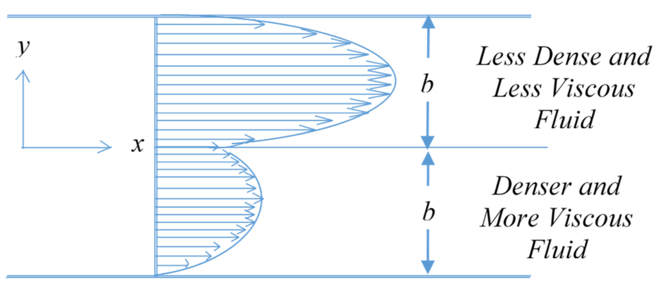

Figure 1: Velocity profile of two incompressible immiscible fluid

The governing equations for the two immiscible compressible fluids are:

$$-\mu_1 \frac{\partial^2 u_1}{\partial x^2} = f_0 \tag{1}$$

$$-\mu_2 \frac{\partial^2 u_2}{\partial x^2} = f_0 \tag{2}$$

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where u_1 and u_2 = Velocities of fluid I and fluid 2 respectively

μ_1 and μ_2 = Viscocity of fluid 1 and fluid 2 respectively

f_0 = Pressure gradient of the fluids

The boundary conditions are:

$$u_1(-b)=0, \quad u_2(b)=0 \quad \text{and} \quad u_1(0)=u_2(0) \tag{3}$$

The domain of the problem consists of all points between *x=0* and *x=L* i.e. $\Omega = (0, L)$. The domain was divided into a set of line elements, a typical element being of length h_e and located between two end points A and B of a typical element. The collection of such elements is called the finite element mesh of the domain. The reason for dividing the domain into finite elements was to represent the geometry of the domain and to approximate the solution over the entire domain.

2.1 Mathematical Analysis

In the development of the weak form, a linear mesh was assumed and placed over the domain. This was done by multiplying Equation 1 by the weighted function (w) and integrating the final equation over the domain. This resulted in the mathematical expression in Equation 4.

$$\int_{x_A}^{x_B} \mu \frac{\partial w}{\partial x}\frac{\partial u}{\partial x}dx - \int_{x_A}^{x_B} f_0 w dx - wQ_A - wQ_B = 0 \tag{4}$$

But $x_B=x_A+h_e$

Where x_A is the initial position of a typical element, x_B is the final position of a typical element and h_e length of a typical element.

Equation 4 is known as the weak form of the governing equation. The weak form requires that the approximation chosen for *u* should be at least linear in *x* so that there are no terms in Equation 4 that are identically zero. Since the primary variable is simply the function itself, the Lagrange family of interpolation functions is admissible. It was proposed that *u* is the approximation over the typical finite element domain by the expression:

$$u = \sum_{j=1}^{n} u_j^e \psi_j^e(x) \text{ and } w = \psi_i^e(x) \qquad i, j = 1, 2, 3 \tag{5}$$

Where $w = \psi_i^e(x)$ is the trial function

In Galerkin's weighted residual method, the weighting functions are chosen to be identical to the trial functions (Reddy, 1993).

Substituting Equation 5 into Equation 4, results in:

$$\mu\left[K_{ij}^{e}\right]\left\{u_{j}^{e}\right\}=f_{0}\left\{F_{i}^{e}\right\}+\left\{Q_{i}^{e}\right\} \tag{6}$$

where

$$K_{ij}^{e}=\int_{x_A}^{x_A+h_e}\frac{\partial\psi_{i}^{e}(x)}{\partial x}\frac{\partial\psi_{j}^{e}(x)}{\partial x}dx \tag{7}$$

$$F_{i}^{e}=\int_{x_A}^{x_A+h_e}\psi_{i}^{e}(x)\,dx \tag{8}$$

$$Q_{i}^{e}=\psi_{i}^{e}(x)Q_{A}+\psi_{i}^{e}(x)Q_{B} \tag{9}$$

Equation 6 is referred to as the finite element-based model while Equation 7 is known as the stiffness matrix and Equation 8 is referred to as the flux matrix. Hence, the one-dimensional Lagrange quadratic interpolation function becomes:

$$\psi_{1}=\left(1-\frac{x}{h_e}\right)\left(1-\frac{2x}{h_e}\right) \tag{10}$$

$$\psi_{2}=\frac{4x}{h_e}\left(1-\frac{x}{h_e}\right) \tag{11}$$

$$\psi_{3}=-\frac{x}{h_e}\left(1-\frac{2x}{h_e}\right) \tag{12}$$

where h_e= Elemental length

To evaluate the K_{ij} matrix, Equations 10-12 were substituted accordingly into Equations 7 and 8 respectively to give Equations 13 and 14.

$$K^{e}=\frac{1}{3h_e^3}\begin{bmatrix} 7h_e^2-24h_ex_A+48x_A^2 & -8\left(h_e^2-3h_ex_A+12x_A^2\right) & h_e^2+48x_A^2 \\ -8\left(h_e^2-3h_ex_A+12x_A^2\right) & 16\left(h_e^2+12x_A^2\right) & -8\left(h_e^2+3h_ex_A+12x_A^2\right) \\ h_e^2+48x_A^2 & -8\left(h_e^2+3h_ex_A+12x_A^2\right) & 7h_e^2+24h_ex_A+48x_A^2 \end{bmatrix} \tag{13}$$

$$F^{e}=\begin{Bmatrix} \dfrac{h_e}{6}-x_A+\dfrac{2x_A^2}{h_e} \\ \dfrac{2\left(h_e^2-6x_A^2\right)}{3h_e} \\ \dfrac{h_e}{6}+x_A+\dfrac{2x_A^2}{h_e} \end{Bmatrix} \tag{14}$$

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Equation 13 represents the generalized form of the stiffness matrix for the entire domain of the fluids between stationary parallel plates and Equation 14 represents the generalized form of the flux matrix for the entire domain of the fluid between stationary parallel plates. In this work, the domain of the parallel plates was divided into four quadratic elements thus:

$$\left[K^e\right]=\frac{1}{3h_e}\begin{bmatrix} 7\mu_1 & -8\mu_1 & \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -8\mu_1 & 16\mu_1 & -8\mu_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mu_1 & -8\mu_1 & 14\mu_1 & -8\mu_1 & \mu_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -8\mu_1 & 16\mu_1 & -8\mu_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_1 & -8\mu_1 & 7\mu_1+7\mu_2 & -8\mu_2 & \mu_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -8\mu_2 & 16\mu_2 & -8\mu_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_2 & -8\mu_2 & 14\mu_2 & -8\mu_2 & \mu_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -8\mu_2 & 16\mu_2 & -8\mu_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mu_2 & -8\mu_2 & 7\mu_2 \end{bmatrix} \tag{15}$$

$$\left\{F^e\right\}=\frac{h_e}{6}\begin{Bmatrix} 1 \\ 4 \\ 2 \\ 4 \\ 2 \\ 4 \\ 2 \\ 4 \\ 1 \end{Bmatrix} \tag{16}$$

Substituting Equations 15 and 16 Equation 6, results in:

$$\begin{bmatrix} 7\mu_1 & -8\mu_1 & \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -8\mu_1 & 16\mu_1 & -8\mu_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mu_1 & -8\mu_1 & 14\mu_1 & -8\mu_1 & \mu_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -8\mu_1 & 16\mu_1 & -8\mu_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_1 & -8\mu_1 & 7\mu_1+7\mu_2 & -8\mu_2 & \mu_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -8\mu_2 & 16\mu_2 & -8\mu_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_2 & -8\mu_2 & 14\mu_2 & -8\mu_2 & \mu_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -8\mu_2 & 16\mu_2 & -8\mu_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mu_2 & -8\mu_2 & 7\mu_2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{Bmatrix}=\frac{h_e^2 f_0}{2}\begin{Bmatrix} 1 \\ 4 \\ 2 \\ 4 \\ 2 \\ 4 \\ 2 \\ 4 \\ 1 \end{Bmatrix}+\begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1+Q_1^2 \\ Q_2^2 \\ Q_3^2+Q_1^3 \\ Q_2^3 \\ Q_3^3+Q_1^4 \\ Q_2^4 \\ Q_3^4 \end{Bmatrix} \tag{17}$$

Due to balance of internal flux, Q_2^1, $Q_3^1+Q_1^2$, Q_2^2, $Q_3^2+Q_1^3$, Q_2^3, $Q_3^3+Q_1^4$, Q_2^4 are equal to zero. Introducing the boundary conditions stated in Equation 3, Equation 17 reduces to:

$$\begin{bmatrix} 16\mu_1 & -8\mu_1 & 0 & 0 & 0 & 0 & 0 \\ -8\mu_1 & 14\mu_1 & -8\mu_1 & \mu_1 & 0 & 0 & 0 \\ 0 & -8\mu_1 & 16\mu_1 & -8\mu_1 & 0 & 0 & 0 \\ 0 & \mu_1 & -8\mu_1 & 7\mu_1+7\mu_2 & -8\mu_2 & \mu_2 & 0 \\ 0 & 0 & 0 & -8\mu_2 & 16\mu_2 & -8\mu_2 & 0 \\ 0 & 0 & 0 & \mu_2 & -8\mu_2 & 14\mu_2 & -8\mu_2 \\ 0 & 0 & 0 & 0 & 0 & -8\mu_2 & 16\mu_2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{Bmatrix} = \frac{h_e^2 f_0}{2} \begin{Bmatrix} 4 \\ 2 \\ 4 \\ 2 \\ 4 \\ 2 \\ 4 \end{Bmatrix} \quad (18)$$

3. RESULTS AND DISCUSSION

The data used in this work are given thus:

$$b = 5m,\ f_0 = 0.3,\ \mu_1 = 0.9Pa.s,\ \mu_2 = 0.1Pa.s,\ h_e = \frac{b}{2}$$

The exact differential equation solution of the problem is given in Equation 19

$$u_i = \frac{f_0 b^2}{2\mu_i}\left[\frac{2\mu_i}{\mu_1+\mu_2} + \frac{\mu_1-\mu_2}{\mu_1+\mu_2}\frac{y}{b} - \left(\frac{y}{b}\right)^2\right] \qquad (i = 1, 2) \quad (19)$$

A graph of the velocity profile of the two immiscible incompressible fluids in between the two stationary parallel plates is as shown in Figure 2. The graph shows the velocities at different nodes plotted against the length of the stationary parallel plates. The graph shows a parabolic relationship between the velocity and the length of the stationary parallel plates. It was observed that the velocities at points -5m and 5m which are the boundaries of the stationary parallel plates are 0 m/s. This was due to the fact that the no slip boundary condition at the boundaries (walls) of the two stationary parallel plates was applied. From Figure 2, between 0 and 5m represents the velocity profile for the less viscous and less dense fluid and between 0 and -5m represents the velocity profile for the more viscous and denser fluid. It is observed from Figure 2 that at point zero, which is at the point of intersection of the two fluids, their velocities are the same with a value of 7.5m/s. From this analysis, the maximum velocity was attained in the less dense region with the lowest viscosity of 0.1Pa.s and the maximum velocity attained was 13.125m/s at a point half way into the less viscous and less dense region i.e., 2.5m from the centre line. It is worthy to note also that if the fluids in between the stationary parallel plates are of the same viscosities of 0.1Pa.s, the maximum velocity of 37.5m/s would have been obtained at the zero point (centre line). Also, if the fluids in between the two stationary parallel plates are of the same viscosities of 0.9Pa.s, the maximum velocity of 4.167m/s would have been obtained at the zero point (centre line). The reason for obtaining the maximum velocity of 13.125m/s at a point half way into the less viscous and less dense region i.e., 2.5m from the centre line when the two fluids have different viscosities was that the more viscous fluid will tend to cause a dragging effect on the less dense fluid, thereby reducing the velocity while the less dense fluid will tend to help the more viscous fluid increase its velocity.

To verify the accuracy of the results obtained from the Finite Element Method, the same problem was solved using the exact differential equation method. The results obtained from the Finite Element Method were compared with the results obtained from the exact differential equation method. It was observed from the two methods that their results were in good agreement with one another. From the results shown in Table 1, even with just four linear element, a very high accuracy was obtained as reflected in the percentage error estimation. The advantage of the FEM over the exact differential equation method is that the FEM gives results that represent the velocities at different nodes for the

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whole material under consideration at the same time unlike the result from the exact differential equation method that provide discrete result at a time and needs further iteration to determine the velocity values at other points of the two stationary parallel plates.

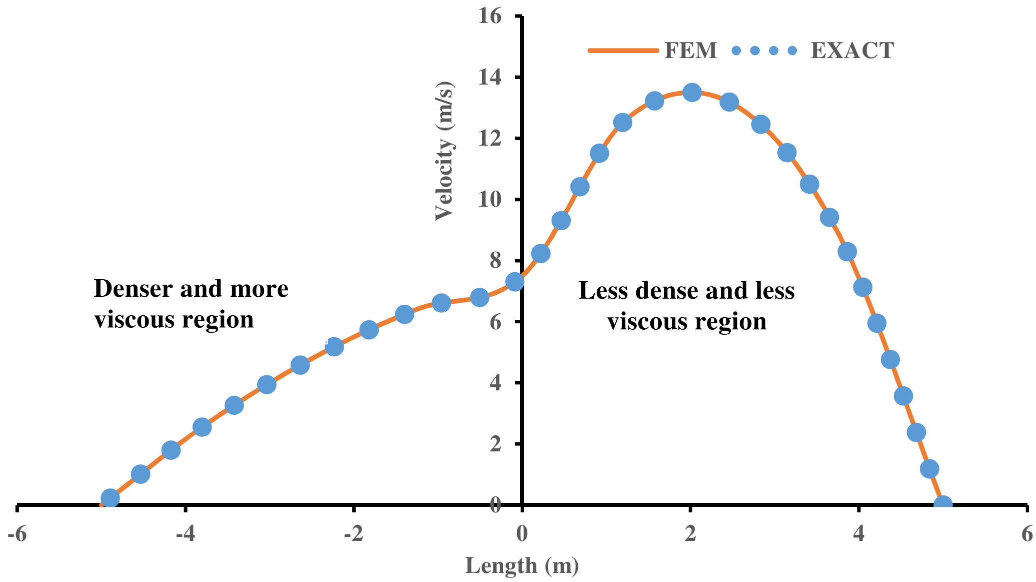

Figure 2: A graph of velocity against length of plate

Table 1: Comparison between the Exact Solution and the FEM Solution

Length (m)	Velocity (m/s)		% Error
	FEM	Exact	
5.00	0.0000	0.0000	0.0000
3.75	8.9063	8.9063	1.99E-14
2.50	13.1250	13.1250	2.71E-14
1.25	12.6563	12.6563	0.0000
0.00	7.5000	7.5000	-2.4E-14
-1.25	6.4063	6.4063	-2.8E-14
-2.50	4.7917	4.7917	-3.7E-14
-3.75	2.6563	2.6563	-3.3E-14
-5.00	0.0000	0.0000	0.0000

4. CONCLUSION

The finite element method has been used to obtain the velocity distribution of two immiscible incompressible fluids with steady laminar flow. The results obtained from the FEM were compared with the results obtained from the exact differential equation method and it was discovered that both results agree. The result obtained shows that the finite element method is an efficient and accurate method.

5. CONFLICT OF INTEREST

There is no conflict of interest associated with this work.

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