Nigerian Research Journal of Engineering and Environmental Sciences
Journal homepage: www.rjees.com

Original Research Article

Evaluation of Partial Differential Equations with Variable Coefficients using He-Laplace Technique

***Usman, S., Saleh, A. and Salisu, M.**
Department of Mathematics, Faculty of Science, Federal University Dutse, P.M.B. 7156, Jigawa State, Nigeria.
*princemagee1@gmail.com

ARTICLE INFORMATION

Article history:
Received 02 October, 2018
Revised 18 October, 2018
Accepted 23 October, 2018
Available online 30 December, 2018

Keywords:
He-Laplace
Heat and wave equations
Analytic solution
RDTM
Partial differential equations

ABSTRACT

In this article, solution to partial differential equations with variable coefficients was presented in different forms using He-Laplace technique. Implementation of this was shown using some numerical examples and compared with known technique. The results show that solutions from the He-Laplace technique is relatively accurate and the technique is more efficient and powerful than the Reduced Differential Transform Method (RDTM) showing the absolute error of the solution converging to zero faster and in less iterations.

1. INTRODUCTION

In mathematics, so many works have been carried out on partial differential equations describing different phenomena such as sound, heat, electrostatics and so on, thus showing that physical problems can be described using mathematical models based on partial differential equations (Narasimhan 1999; Herman 2015). Partial differential equations with variable coefficients have been tackled by numerous numerical methods in order to obtain approximate solutions to the problems and in some cases closed form solutions (exact) obtained.

Adomian decomposition method was used in determining the approximate solution for heat-like and wave-like models with variable coefficients by Wazwaz and Gorguis (2004). The method was demonstrated for some variety of problems in one and higher dimensional spaces where the exact solutions are obtained. The results obtained in all cases show the reliability and the efficiency of the method. Alomari *et al*. (2008) obtained the approximate analytic solution to heat-like and wave-like equations using Homotopy analysis method. The Homotopy analysis method contains an auxiliary parameter $\hbar$ which provides a convenient way of controlling the convergence region of the series solutions. The results obtained proved that Homotopy analysis method is very effective and simple with less error than the Adomian decomposition method and the variation iteration method. Parabolic-like equations and hyperbolic-like equations were solved using

S. Usman et al. / Nigerian Research Journal of Engineering and Environmental Sciences
3(2) 2018 pp. 571-583

Homotopy perturbation method by Jin (2008) where the numerical results showed that the method is a promising and powerful tool for solving partial differential equations with variable coefficients. Also, the method is flexible and able to solve parabolic-like and hyperbolic-like equations. Noor and Mohyud-Din (2008) presented solutions to heat-like and wave-like equations using the modified variational iteration method. The proposed modification is made by introducing He's polynomials in the correction functional and the iteration scheme finds the solution without any discretization, linearization or restrictive assumption. The reduced differential transform method was used in solving various partial differential equations by Keskin and Oturanc (2009) where the demerit of complex calculation of differential transform method is shown. The method shows it is effective and powerful. He's polynomials was employed in tackling heat-like and wave-like equations as presented by Mohyud-Din (2009). It was seen that the proposed iterative scheme finds the solution without any discretization, linearization, or restrictive assumptions. Reduced differential method was used in solving regularized long wave equation by Keskin and Oturanc (2010a). Also, the reduced differential transformation method was used in solving the gas dynamics equations (Keskin and Oturanc 2010b). Cenesiz *et al.* (2010) presented the solution of nonlinear dispersive k(m,n) type equation using the reduced differential transform method. The generalized Korteweg-de Vries equations were solved using the reduced differential transform method by Keskin and Oturanc (2010c). Bushra (2011) solved partial differential equations in different dimensions using reduced differential transform method. Benhammouda *et al.* (2014) applied reduced differential transform method (RDTM) in finding the solution of partial differential-algebraic equations (PDAEs). Also, it was shown that two systems of index-two and index-three are solved to show that RDTM can provide analytical solutions for PDAEs in convergent series form also the technique is based on a straightforward step and does not generate secular term or depend on a perturbation parameter.

From all these literatures, the He-Laplace technique has not been used in solving heat-like and wave-like equations and which forms the basis for our work. This work is thus aimed at solving heat-like and wave-like equations using He-Laplace technique and then comparing result with already established result and also checking the efficiency of the methods.

2. METHODOLOGY

He-Laplace technique is a combination of Homotopy perturbation method and the Laplace transformation method used in solving linear and nonlinear partial differential equations (Hradyesh and Atulya 2012). To illustrate the basics idea of this technique, consider a general nonlinear nonhomogeneous partial differential equation with initial conditions of the form:

$$\frac{\partial^2 y}{\partial t^2} + R_1(x,t) + R_2 y(x,t) + Ny(x,t) = f(x,t),\ y(x,0) = \alpha(x),\ \frac{\partial y}{\partial t}(x,0) = \beta(x) \quad (1)$$

Where $R_1 = \frac{\partial^2}{\partial x^2}\ and\ R_2 = \frac{\partial}{\partial x}$ are the linear differential operators, N represents the general nonlinear differential operator and $f(x,t)$ the source term. Taking Laplace transform and then inverse Laplace transform of Equation (1) gives:

$$y(x,t) = F(x,t) - L^{-1}[\frac{1}{s^2}\{L[R_1 y(x,t) + R_2 y(x,t)] + L[Ny(x,t)]\}] \quad (2)$$

where L is the Laplace operator, *F(x,t)* the term rising from the source term and the prescribed initial conditions.

Applying the Homotopy perturbation technique:

$$y(x,t) = \sum_{n=0}^{\infty} p^n y_n(x,t) \quad (3)$$

S. Usman et al. / Nigerian Research Journal of Engineering and Environmental Sciences
3(2) 2018 pp. 571-583

And the nonlinear term:

$$Ny(x,t) = \sum_{n=0}^{\infty} p^n H_n(y) \quad (4)$$

The combination of Laplace transform and Homotopy perturbation technique gives:

$$\sum_{n=0}^{\infty} p^n y_n(x,t) = F(x,t) - p(L^{-1}\{\frac{1}{s^2} L[(R_1 + R_2)\sum_{n=0}^{\infty} p^n y_n(x,t) + \sum_{n=0}^{\infty} p^n H_n(y)]\}) \quad (5)$$

Comparing the coefficients of like powers of p, gives the following approximations:

$$\begin{aligned} p^0 &: y_0(x,t) = F(x,t) \\ p^1 &: y_1(x,t) = -L^{-1}\{\frac{1}{s^2} L[(R_1 + R_2) y_0(x,t) + H_0(y)]\} \\ p^2 &: y_2(x,t) = -L^{-1}\{\frac{1}{s^2} L[(R_1 + R_2) y_1(x,t) + H_1(y)]\} \\ p^3 &: y_3(x,t) = -L^{-1}\{\frac{1}{s^2} L[(R_1 + R_2) y_2(x,t) + H_2(y)]\} \end{aligned} \quad (6)$$

.
.
.

Hence the general solution takes the form:

$$y(x,t) = y_0(x,t) + y_1(x,t) + y_2(x,t) + y_3(x,t) + \cdots \quad (7)$$

3. RESULTS AND DISCUSSION

Using the He-Laplace technique in solving the heat-like and wave-like equations is thus illustrated.

Example 1:

Considering the heat-like equation which is a one-dimensional initial value problem.

$$\frac{\partial u}{\partial t} = \frac{x^2}{2}\frac{\partial^2 u}{\partial x^2} \quad (8)$$

With initial condition:

$$u(x,0) = x^2 \quad (9)$$

Taking the Laplace transform of Equation (8) and applying the initial condition (Equation 9), gives:

$$L\{u(x,s)\} - \frac{x^2}{s} = \frac{1}{s} L\{\frac{x^2}{2}\frac{\partial^2 u}{\partial x^2}\} \quad (10)$$

The Laplace transform inverse yields:

$$u(x,t) = x^2 + L^{-1}\{\frac{1}{s} L\{\frac{x^2}{2}\frac{\partial^2 u}{\partial x^2}\}\} \quad (11)$$

Applying the Homotopy perturbation method yields:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = x^2 + p(L^{-1}\{\frac{1}{s}L\{\frac{x^2}{2}\frac{\partial^2 u}{\partial x^2}\}\}) \quad (12)$$

Comparing the coefficient of like powers of p, gives:

$$p^0: u_0(x,t) = x^2 \quad (13)$$
$$p^1: u_1(x,t) = L^{-1}\left\{\frac{1}{s}L\left\{\frac{x^2}{2}\frac{\partial^2 u_0}{\partial x^2}\right\}\right\} = x^2 t \quad (14)$$
$$p^2: u_2(x,t) = L^{-1}\left\{\frac{1}{s}L\left\{\frac{x^2}{2}\frac{\partial^2 u_1}{\partial x^2}\right\}\right\} = \frac{x^2t^2}{2!} \quad (15)$$
$$p^3: u_3(x,t) = L^{-1}\left\{\frac{1}{s}L\left\{\frac{x^2}{2}\frac{\partial^2 u_2}{\partial x^2}\right\}\right\} = \frac{x^2t^3}{3!} \quad (16)$$
$$p^4: u_4(x,t) = L^{-1}\left\{\frac{1}{s}L\left\{\frac{x^2}{2}\frac{\partial^2 u_3}{\partial x^2}\right\}\right\} = \frac{x^2t^4}{4!} \quad (17)$$

.
.
.

The solution $u(x,t)$ is given by:

$$u(x,t) = u_0 + u_1 + u_2 + u_3 + u_4 + \cdots$$
$$= x^2[1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots] \quad (18)$$

Also, the closed form solution of Equation (18) becomes:

$$u(x,t) = x^2 e^t \quad (19)$$

Table 1 shows the comparison between the approximate solution obtained by the Reduced Differential Transform Method in the work of Bushra (2011) with that obtained by He-Laplace Technique. Considering only the first five iterations for both methods, it was observed that for the same number of terms the same result was obtained which indicates that the two techniques are in agreement.

Table 1: Absolute error for example 1 at $t = 0.1$

x	$\lvert u(x,t)_{exact} - u_5(x,t)\rvert$	
	RDTM	He-Laplace method
0	0.0	0.0
1	8.5×10^{-8}	8.5×10^{-8}
2	3.39×10^{-7}	3.39×10^{-7}
3	7.63×10^{-7}	7.63×10^{-7}
4	1.35×10^{-6}	1.35×10^{-6}
5	2.12×10^{-6}	2.12×10^{-6}

Example 2:

The heat-like (Equation 20) is a two-dimensional initial value problem.

$$\frac{\partial u}{\partial t} = \frac{y^2}{2}\frac{\partial^2 u}{\partial x^2} + \frac{x^2}{2}\frac{\partial^2 u}{\partial y^2} \quad (20)$$

With initial condition:

$$u(x,y,0) = y^2 \tag{21}$$

Taking the Laplace transform of Equation (20) and applying the initial condition (Equation 21) yields:

$$L\{u(x,y,s)\} = \frac{y^2}{s} + \frac{y^2}{2s} L\left\{\frac{\partial^2 u}{\partial x^2}\right\} + \frac{x^2}{2s} L\left\{\frac{\partial^2 u}{\partial y^2}\right\} \tag{22}$$

The Laplace transform inverse gives:

$$u(x,y,t) = y^2 + L^{-1}\left\{\frac{y^2}{2s} L\left\{\frac{\partial^2 u}{\partial x^2}\right\}\right\} + L^{-1}\left\{\frac{x^2}{2s} L\left\{\frac{\partial^2 u}{\partial y^2}\right\}\right\} \tag{23}$$

Applying the Homotopy perturbation method, gives:

$$\sum_{n=0}^{\infty} p^n u^n = y^2 + p(L^{-1}\left\{\frac{y^2}{2s} L\left\{\frac{\partial^2 u}{\partial x^2}\right\}\right\} + L^{-1}\left\{\frac{x^2}{2s} L\left\{\frac{\partial^2 u}{\partial y^2}\right\}\right\}) \tag{24}$$

Comparing the coefficient of like powers of p gives:

$$p^0: u_0(x,y,t) = y^2 \tag{25}$$

$$p^1: u_1(x,y,t) = L^{-1}\left\{\frac{y^2}{2s} L\left\{\frac{\partial^2 u_0}{\partial x^2}\right\}\right\} + L^{-1}\left\{\frac{x^2}{2s} L\left\{\frac{\partial^2 u_0}{\partial y^2}\right\}\right\} = x^2 t \tag{26}$$

$$p^2: u_2(x,y,t) = L^{-1}\left\{\frac{y^2}{2s} L\left\{\frac{\partial^2 u_1}{\partial x^2}\right\}\right\} + L^{-1}\left\{\frac{x^2}{2s} L\left\{\frac{\partial^2 u_1}{\partial y^2}\right\}\right\} = \frac{y^2 t^2}{2!} \tag{27}$$

$$p^3: u_3(x,y,t) = L^{-1}\left\{\frac{y^2}{2s} L\left\{\frac{\partial^2 u_2}{\partial x^2}\right\}\right\} + L^{-1}\left\{\frac{x^2}{2s} L\left\{\frac{\partial^2 u_2}{\partial y^2}\right\}\right\} = \frac{x^2 t^3}{3!} \tag{28}$$

$$p^4: u_4(x,y,t) = L^{-1}\left\{\frac{y^2}{2s} L\left\{\frac{\partial^2 u_3}{\partial x^2}\right\}\right\} + L^{-1}\left\{\frac{x^2}{2s} L\left\{\frac{\partial^2 u_3}{\partial y^2}\right\}\right\} = \frac{y^2 t^4}{4!} \tag{29}$$

$$p^5: u_5(x,y,t) = L^{-1}\left\{\frac{y^2}{2s} L\left\{\frac{\partial^2 u_4}{\partial x^2}\right\}\right\} + L^{-1}\left\{\frac{x^2}{2s} L\left\{\frac{\partial^2 u_4}{\partial y^2}\right\}\right\} = \frac{x^2 t^5}{5!} \tag{30}$$

.
.
.

The solution $u(x,y,t)$ is given by:

$$u(x,y,t) = u_0 + u_1 + u_2 + u_3 + u_4 + u_5 + \cdots$$

$$= y^2\left[1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots\right] + x^2\left[t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\right] \tag{31}$$

The closed form solution of Equation (31) becomes:

$$u(x,y,t) = y^2\cosh(t) + x^2\sinh(t) \tag{32}$$

Comparison of the approximate solution obtained by Reduced Differential Transform Method in the work of Bushra (2011) with that obtained by He-Laplace Technique is shown in Table 2. Considering only the

S. Usman et al. / Nigerian Research Journal of Engineering and Environmental Sciences
3(2) 2018 pp. 571-583

first six iterations for both methods, we see also that for the same number of terms the same result was obtained showing that the two techniques are in agreement.

Table 2: Absolute error for example.2 at $t = 0.1$

x	y	$\lvert u(x,t)_{exact} - u_6(x,t)\rvert$	
		RDTM	He-Laplace method
0	0	0.0	0.0
1	1	1×10^{-9}	1×10^{-9}
2	2	4×10^{-9}	4×10^{-9}
3	3	9×10^{-9}	9×10^{-9}
4	4	1.8×10^{-8}	1.8×10^{-8}
5	5	2.5×10^{-8}	2.5×10^{-8}

Example 3:

Consider a three-dimensional initial value problem which describe the heat-like equation;

$$\frac{\partial u}{\partial t} = (xyz)^4 + \frac{1}{36}[x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + z^2\frac{\partial^2 u}{\partial z^2}] \quad (33)$$

With initial condition

$$u(x,y,z,0) = 0 \quad (34)$$

Taking the Laplace transform of Equation (33) and applying the initial condition (Equation 34) gives:

$$L\{u(x,y,z,s)\} = \frac{1}{s}L\{(xyz)^4\} + \frac{1}{s}L\{\frac{1}{36}[x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + z^2\frac{\partial^2 u}{\partial z^2}]\} \quad (35)$$

The Laplace transform inverse yields:

$$u(x,y,z,t) = (xyz)^4 t + L^{-1}\{\frac{1}{s}L\{\frac{1}{36}\left[x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + z^2\frac{\partial^2 u}{\partial z^2}\right]\}\} \quad (36)$$

Applying the Homotopy perturbation method, yields:

$$\sum_{n=0}^{\infty} p^n u_n(x,y,z,t) = (xyz)^4 t + p(L^{-1}\{\frac{1}{s}L\left\{\frac{1}{36}\left[x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + z^2\frac{\partial^2 u}{\partial z^2}\right]\right\}\}) \quad (37)$$

Comparing the coefficient of like powers of p yields:

$$p^0: u_0(x,y,z,t) = (xyz)^4 t \quad (38)$$
$$p^1: u_1(x,y,z,t) = L^{-1}\left\{\frac{1}{s}L\left\{\frac{1}{36}\left[x^2\frac{\partial^2 u_0}{\partial x^2} + y^2\frac{\partial^2 u_0}{\partial y^2} + z^2\frac{\partial^2 u_0}{\partial z^2}\right]\right\}\right\} = \frac{(xyz)^4 t^2}{2!} \quad (39)$$
$$p^2: u_2(x,y,z,t) = L^{-1}\left\{\frac{1}{s}L\left\{\frac{1}{36}\left[x^2\frac{\partial^2 u_1}{\partial x^2} + y^2\frac{\partial^2 u_1}{\partial y^2} + z^2\frac{\partial^2 u_1}{\partial z^2}\right]\right\}\right\} = \frac{(xyz)^4 t^3}{3!} \quad (40)$$
$$p^3: u_3(x,y,z,t) = L^{-1}\left\{\frac{1}{s}L\left\{\frac{1}{36}\left[x^2\frac{\partial^2 u_2}{\partial x^2} + y^2\frac{\partial^2 u_2}{\partial y^2} + z^2\frac{\partial^2 u_2}{\partial z^2}\right]\right\}\right\} = \frac{(xyz)^4 t^4}{4!} \quad (41)$$

$$p^4: u_4(x,y,z,t) = L^{-1}\left\{\frac{1}{s}L\left\{\frac{1}{36}\left[x^2\frac{\partial^2 u_3}{\partial x^2} + y^2\frac{\partial^2 u_3}{\partial y^2} + z^2\frac{\partial^2 u_3}{\partial z^2}\right]\right\}\right\} = \frac{(xyz)^4 t^5}{5!} \tag{42}$$

.
.
.

The solution $u(x,y,z,t)$ is given by:

$$u(x,y,z,t) = u_0 + u_1 + u_2 + u_3 + u_4 + \cdots$$

$$= (xyz)^4\left[t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \cdots\right] \tag{43}$$

The closed form solution of equation (43) becomes

$$u(x,y,z,t) = (xyz)^4[e^t - 1] \tag{44}$$

In Table 3, comparison is shown between the approximate solution obtained by Reduced Differential Transform Method in the work of Bushra (2011) with that obtained by He-Laplace Technique. Considering only the first five iterations for both methods, it is seen that for the same number of terms, the He-Laplace technique gives relatively more accurate results than the other method since the absolute error result for He-Laplace technique tends to zero faster than that of Reduced Differential Transform Method.

Table 3: Absolute error for example 3 at $t = 0.1$

x	y	z	$\lvert u(x,t)_{exact} - u_5(x,t)\rvert$	
			RDTM	He-Laplace method
0	0	0	0.0	0.0
1	1	1	8.5×10^{-8}	2×10^{-9}
2	2	2	3.48×10^{-4}	8.2×10^{-6}
3	3	3	4.51×10^{-2}	1.06×10^{-3}
4	4	4	1.42	3.3×10^{-2}
5	5	5	20.75	4.9×10^{-1}

Example 4:

Consider the wave-like equation which is a one-dimensional initial value problem.

$$\frac{\partial^2 u}{\partial t^2} = \frac{x^2}{2}\frac{\partial^2 u}{\partial x^2} \tag{45}$$

With initial condition

$$u(x,0) = x,\ \frac{\partial u}{\partial t}(x,0) = x^2 \tag{46}$$

Taking the Laplace transform of Equation (45) and applying the initial condition (Equation 46) gives:

$$L\{u(x,s)\} = \frac{x}{s} + \frac{x^2}{s^2} + \frac{1}{s^2}L\{\frac{x^2}{2}\frac{\partial^2 u}{\partial x^2}\} \tag{47}$$

The Laplace transform inverse gives:

$$u(x,t) = x + x^2t + L^{-1}\{\frac{1}{s^2}L\{\frac{x^2}{2}\frac{\partial^2 u}{\partial x^2}\}\} \tag{48}$$

Applying the Homotopy perturbation method, yields:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = x + x^2t + p(L^{-1}\{\frac{1}{s^2}L\{\frac{x^2}{2}\frac{\partial^2 u}{\partial x^2}\}\}) \tag{49}$$

Comparing the coefficient of like powers of p yields:

$$p^0: u_0(x,t) = x + x^2t \tag{50}$$

$$p^1: u_1(x,t) = L^{-1}\left\{\frac{1}{s^2}L\left\{\frac{x^2}{2}\frac{\partial^2 u_0}{\partial x^2}\right\}\right\} = \frac{x^2t^3}{3!} \tag{51}$$

$$p^2: u_2(x,t) = L^{-1}\left\{\frac{1}{s^2}L\left\{\frac{x^2}{2}\frac{\partial^2 u_1}{\partial x^2}\right\}\right\} = \frac{x^2t^5}{5!} \tag{52}$$

$$p^3: u_3(x,t) = L^{-1}\left\{\frac{1}{s^2}L\left\{\frac{x^2}{2}\frac{\partial^2 u_2}{\partial x^2}\right\}\right\} = \frac{x^2t^7}{7!} \tag{53}$$

$$p^4: u_4(x,t) = L^{-1}\left\{\frac{1}{s^2}L\left\{\frac{x^2}{2}\frac{\partial^2 u_3}{\partial x^2}\right\}\right\} = \frac{x^2t^9}{9!} \tag{54}$$

$$p^5: u_5(x,t) = L^{-1}\left\{\frac{1}{s^2}L\left\{\frac{x^2}{2}\frac{\partial^2 u_4}{\partial x^2}\right\}\right\} = \frac{x^2t^{11}}{11!} \tag{55}$$

$$p^6: u_6(x,t) = L^{-1}\left\{\frac{1}{s^2}L\left\{\frac{x^2}{2}\frac{\partial^2 u_5}{\partial x^2}\right\}\right\} = \frac{x^2t^{13}}{13!} \tag{56}$$

.
.
.

The solution $u(x,t)$ is given as:

$$u(x,t) = u_0 + u_1 + u_2 + u_3 + u_4 + u_5 + u_6 \ldots$$

$$= x + x^2\left[t + \frac{t^3}{3!} + \frac{t^5}{5!} + \frac{t^7}{7!} + \frac{t^9}{9!} + \frac{t^{11}}{11!} + \frac{t^{13}}{13!} + \cdots\right] \tag{57}$$

The closed form solution of Equation (57) becomes:

$$u(x,t) = x + x^2\sinh(t) \tag{58}$$

Table 4: Absolute error for example 4 at $t = 0.1$

x	$\lvert u(x,t)_{exact} - u_7(x,t)\rvert$	
	RDTM	He-Laplace method
0	0.0	0.0
1	0.0	0.0
2	0.0	0.0
3	0.0	0.0
4	0.0	0.0
5	0.0	0.0

S. Usman et al. / Nigerian Research Journal of Engineering and Environmental Sciences
3(2) 2018 pp. 571-583

Also, comparison between the approximate solution obtained by Reduced Differential Transform Method in the work of Bushra (2011) with that obtained by He-Laplace Technique is shown in Table 4. Considering only the first seven iterations for both methods, it was observed that for the same number of terms, the same results is arrived at, indicating that the two techniques are in agreement.

Example 5:

Consider the wave-like equation which is a two-dimensional initial value problem.

$$\frac{\partial^2 u}{\partial t^2} = \frac{x^2}{12}\frac{\partial^2 u}{\partial x^2} + \frac{y^2}{12}\frac{\partial^2 u}{\partial y^2} \tag{59}$$

With initial condition

$$u(x,y,0) = x^4, \ \ \frac{\partial u}{\partial t}(x,y,0) = y^4 \tag{60}$$

Taking the Laplace transform of Equation (59) and applying the initial condition (Equation 60) yields:

$$L\{u(x,y,s)\} = \frac{x^4}{s} + \frac{y^4}{s^2} + \frac{1}{s^2}L\{\frac{x^2}{12}\frac{\partial^2 u}{\partial x^2} + \frac{y^2}{12}\frac{\partial^2 u}{\partial y^2}\} \tag{61}$$

The Laplace transform inverse gives:

$$u(x,y,t) = x^4 + y^4 t + L^{-1}\{\frac{1}{s^2}L\{\frac{x^2}{12}\frac{\partial^2 u}{\partial x^2} + \frac{y^2}{12}\frac{\partial^2 u}{\partial y^2}\}\} \tag{62}$$

Applying the Homotopy perturbation method, yields:

$$\sum_{n=0}^{\infty} p^n u_n\,(x,y,t) = x^4 + y^4 t + p(L^{-1}\{\frac{1}{s^2}L\{\frac{x^2}{12}\frac{\partial^2 u}{\partial x^2} + \frac{y^2}{12}\frac{\partial^2 u}{\partial y^2}\}\}) \tag{63}$$

So, comparing the coefficient of like powers of p, gives:

$$p^0: u_0(x,y,t) = x^4 + y^4 t \tag{64}$$

$$p^1: u_1(x,y,t) = L^{-1}\left\{\frac{1}{s^2}L\left\{\frac{x^2}{12}\frac{\partial^2 u_0}{\partial x^2} + \frac{y^2}{12}\frac{\partial^2 u_0}{\partial y^2}\right\}\right\} = \frac{x^4 t^2}{2!} + \frac{y^4 t^3}{3!} \tag{65}$$

$$p^2: u_2(x,y,t) = L^{-1}\left\{\frac{1}{s^2}L\left\{\frac{x^2}{12}\frac{\partial^2 u_1}{\partial x^2} + \frac{y^2}{12}\frac{\partial^2 u_1}{\partial y^2}\right\}\right\} = \frac{x^4 t^4}{4!} + \frac{y^4 t^5}{5!} \tag{66}$$

$$p^3: u_3(x,y,t) = L^{-1}\left\{\frac{1}{s^2}L\left\{\frac{x^2}{12}\frac{\partial^2 u_2}{\partial x^2} + \frac{y^2}{12}\frac{\partial^2 u_2}{\partial y^2}\right\}\right\} = \frac{x^4 t^6}{6!} + \frac{y^4 t^7}{7!} \tag{67}$$

$$p^4: u_4(x,y,t) = L^{-1}\left\{\frac{1}{s^2}L\left\{\frac{x^2}{12}\frac{\partial^2 u_3}{\partial x^2} + \frac{y^2}{12}\frac{\partial^2 u_3}{\partial y^2}\right\}\right\} = \frac{x^4 t^8}{8!} + \frac{y^4 t^9}{9!} \tag{68}$$

$$p^5: u_5(x,y,t) = L^{-1}\left\{\frac{1}{s^2}L\left\{\frac{x^2}{12}\frac{\partial^2 u_4}{\partial x^2} + \frac{y^2}{12}\frac{\partial^2 u_4}{\partial y^2}\right\}\right\} = \frac{x^4 t^{10}}{10!} + \frac{y^4 t^{11}}{11!} \tag{69}$$

.
.
.

So that the solution $u(x,y,t)$ is given by:

S. Usman et al. / Nigerian Research Journal of Engineering and Environmental Sciences
3(2) 2018 pp. 571-583

$$u(x,y,t) = u_0 + u_1 + u_2 + u_3 + u_4 + u_5 + \cdots$$

$$= x^4\left[1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \frac{t^8}{8!} + \frac{t^{10}}{10!} + \cdots\right] + y^4\left[t + \frac{t^3}{3!} + \frac{t^5}{5!} + \frac{t^7}{7!} + \frac{t^9}{9!} + \frac{t^{11}}{11!} + \cdots\right] \quad (70)$$

The closed form solution of Equation (70) becomes:

$$u(x,y,t) = x^4\cosh(t) + y^4\sinh(t) \quad (71)$$

Table 5 shows the comparison between the approximate solution obtained by Reduced Differential Transform Method in the work of Bushra (2011) with that obtained by He-Laplace Technique. Considering only the first six iterations for both methods, it was observed that for the same number of terms, the He-Laplace technique gives relatively more accurate results than the other method (RDTM).

Table 5: Absolute error for example 5 at $t = 0.1$

x	y	$\lvert u(x,t)_{exact} - u_6(x,t)\rvert$	
		RDTM	He-Laplace method
0	0	0.0	0.0
1	1	1×10^{-9}	0.0
2	2	1.8×10^{-8}	0.0
3	3	8.3×10^{-8}	0.0
4	4	2.48×10^{-7}	0.0
5	5	6.75×10^{-7}	0.0

Example 6:

Equation (72) is a three-dimensional initial value problem describing a wave-like equation.

$$\frac{\partial^2 u}{\partial t^2} = (x^2 + y^2 + z^2) + \frac{1}{2}[x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + z^2\frac{\partial^2 u}{\partial z^2}] \quad (72)$$

With initial condition

$$u(x,y,z,0) = 0,\ \frac{\partial u}{\partial t}(x,y,z,0) = x^2 + y^2 - z^2 \quad (73)$$

Taking the Laplace transform of Equation (72) and applying the initial condition (Equation 73) yields:

$$L\{u(x,y,z,s)\} = \frac{x^2+y^2-z^2}{s^2} + \frac{1}{s^2}L\{(x^2 + y^2 + z^2)\} + \frac{1}{s^2}L\{\frac{1}{2}[x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + z^2\frac{\partial^2 u}{\partial z^2}]\} \quad (74)$$

The Laplace transform inverse yields:

$$u(x,y,z,t) = (x^2 + y^2 - z^2)t + \frac{(x^2+y^2+z^2)t^2}{2!} + L^{-1}\{\frac{1}{s^2}L\{\frac{1}{2}[x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + z^2\frac{\partial^2 u}{\partial z^2}]\}\} \quad (75)$$

Applying the Homotopy perturbation method, yields:

$$\sum_{n=0}^{\infty} p^n u_n(x,y,z,t) = (x^2 + y^2 - z^2)t + \frac{(x^2+y^2+z^2)t^2}{2!} + p(L^{-1}\left\{\frac{1}{s^2}L\left\{\frac{1}{2}\left[x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + z^2\frac{\partial^2 u}{\partial z^2}\right]\right\}\right\}) \quad (76)$$

S. Usman et al. / Nigerian Research Journal of Engineering and Environmental Sciences
3(2) 2018 pp. 571-583

Comparing the coefficient of like powers of p gives:

$$p^0: u_0(x,y,z,t) = (x^2+y^2-z^2)t + \frac{(x^2+y^2+z^2)t^2}{2!} \tag{77}$$

$$p^1: u_1(x,y,z,t) = L^{-1}\left\{\frac{1}{s^2}L\left\{\frac{1}{2}\left[x^2\frac{\partial^2 u_0}{\partial x^2} + y^2\frac{\partial^2 u_0}{\partial y^2} + z^2\frac{\partial^2 u_0}{\partial z^2}\right]\right\}\right\}$$
$$= \frac{(x^2+y^2-z^2)t^3}{3!} + \frac{(x^2+y^2+z^2)t^4}{4!} \tag{78}$$

$$p^2: u_2(x,y,z,t) = L^{-1}\left\{\frac{1}{s^2}L\left\{\frac{1}{2}\left[x^2\frac{\partial^2 u_1}{\partial x^2} + y^2\frac{\partial^2 u_1}{\partial y^2} + z^2\frac{\partial^2 u_1}{\partial z^2}\right]\right\}\right\}$$
$$= \frac{(x^2+y^2-z^2)t^5}{5!} + \frac{(x^2+y^2+z^2)t^6}{6!} \tag{79}$$

$$p^3: u_3(x,y,z,t) = L^{-1}\left\{\frac{1}{s^2}L\left\{\frac{1}{2}\left[x^2\frac{\partial^2 u_2}{\partial x^2} + y^2\frac{\partial^2 u_2}{\partial y^2} + z^2\frac{\partial^2 u_2}{\partial z^2}\right]\right\}\right\}$$
$$= \frac{(x^2+y^2-z^2)t^7}{7!} + \frac{(x^2+y^2+z^2)t^8}{8!} \tag{80}$$

$$p^4: u_4(x,y,z,t) = L^{-1}\left\{\frac{1}{s^2}L\left\{\frac{1}{2}\left[x^2\frac{\partial^2 u_3}{\partial x^2} + y^2\frac{\partial^2 u_3}{\partial y^2} + z^2\frac{\partial^2 u_3}{\partial z^2}\right]\right\}\right\}$$
$$= \frac{(x^2+y^2-z^2)t^9}{9!} + \frac{(x^2+y^2+z^2)t^{10}}{10!} \tag{81}$$

$$p^5: u_5(x,y,z,t) = L^{-1}\left\{\frac{1}{s^2}L\left\{\frac{1}{2}\left[x^2\frac{\partial^2 u_4}{\partial x^2} + y^2\frac{\partial^2 u_4}{\partial y^2} + z^2\frac{\partial^2 u_4}{\partial z^2}\right]\right\}\right\}$$
$$= \frac{(x^2+y^2-z^2)t^{11}}{11!} + \frac{(x^2+y^2+z^2)t^{12}}{12!} \tag{82}$$

.
.
.

The solution $u(x,y,z,t)$ is given as:

$$u(x,y,z,t) = u_0 + u_1 + u_2 + u_3 + u_4 + u_5 + \cdots$$

$$= (x^2+y^2-z^2)\left[t + \frac{t^3}{3!} + \frac{t^5}{5!} + \frac{t^7}{7!} + \frac{t^9}{9!} + \frac{t^{11}}{11!} + \cdots\right] + (x^2+y^2+z^2)\left[\frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \frac{t^8}{8!} + \frac{t^{10}}{10!} + \frac{t^{12}}{12!} + \cdots\right] \tag{83}$$

The closed form solution of Equation (83) becomes:

$$u(x,y,z,t) = (x^2+y^2-z^2)\sinh(t) + (x^2+y^2+z^2)[\cosh(t) - 1] \tag{84}$$

Table 6: Absolute error for example 6 at $t = 0.1$

x	y	z	$\lvert u(x,t)_{exact} - u_6(x,t)\rvert$	
			RDTM	He-Laplace method
0	0	0	0.0	0.0
1	1	1	4×10^{-9}	0.0
2	2	2	1.6×10^{-8}	0.0
3	3	3	3.6×10^{-8}	2×10^{-9}
4	4	4	6.4×10^{-8}	3×10^{-9}
5	5	5	1×10^{-7}	4×10^{-9}

Table 6 gives the comparison also between the approximate solution obtained by the Reduced Differential transform Method in the work of Bushra (2011) and the solution obtained by He-Laplace Technique. Looking at the first six iterations for both methods, it was observed that for the same number of terms the absolute error results for the He-Laplace Technique tends to zero faster than that of the Reduced Differential Transform Method indicating that He-Laplace Technique provides relatively more accurate results than the latter.

4. CONCLUSION

In this paper, He-Laplace technique was applied in obtaining approximate solutions to heat-like and wave-like equations. In addition, the performance of He-Laplace technique with that of Reduced Differential Transform method was compared. Some examples were shown where for the same number of terms He-Laplace method presents a better and relatively more accurate result than Reduced Differential Transform method.

5. ACKNOWLEDGMENT

The authors wish to acknowledge the assistance of the non-academic staff of Department of Mathematics, Federal University Dutse, Dutse toward the success of this work.

6. CONFLICT OF INTEREST

There is no conflict of interest associated with this work.

REFERENCES

Alomari, A. K., Noorani, M. S. M. and Nazari, R. (2008). Solution of Heat-Like and Wave-Like Equations with Variable coefficients by Means of the Homotopy Analysis Method. *Chinese Physics Letters*, 25(2), pp. 589-593.

Benhammouda, B., Vazquez-Leal, H. and Sarmiento-Reyes, A. (2014). Modified Reduced Differential Transform Method for Partial Differential-Algebraic Equations. *Journal of Applied Mathematics*, 2014, pp. 1-9.

Bushra, A. T. (2011). The Use of Reduced Differential Transform Method for Solving Partial Differential Equations with Variable Coefficients. *Journal of Basrah Researchers (Science)*, 37(4), pp. 226-233.

Cenesiz, Y., Keskin, Y., and Kurnaz, A. (2010). The Solution of the Nonlinear Dispersive k(m,n) Equations by Reduced Differential Transform Method. *International Journal of Nonlinear Sciences*, 9(4), pp. 461-467.

Herman, R. L. (2015). Introduction to Partial Differential Equations.

Hradyesh, K. M. and Atulya, K. N. (2012). He- laplace method for linear and nonlinear partial differential equations. *Journal of applied mathematics*, 2012, p.16.

Jin, L. (2008). Homotopy Perturbation Method for Solving Partial Differential Equations with Variable Coefficients. *International Journal of Contemporary Mathematical Sciences*, 3(28), pp. 1395-1407.

Keskin, Y. and Oturanc, G. (2009). Reduced Differential Transform Method for Partial Differential Equation. *International Journal of Nonlinear Sciences and Numerical Simulation*, 10(6), pp. 741-749.

Keskin, Y. and Oturanc, G. (2010a). Numerical Solution of Regularized Long Wave Equation by Reduced Differential Transform Method. *Applied Mathematical Sciences*, 4(25), pp. 1221-1231.

Keskin, Y. and Oturanc, G. (2010b). Application of Reduced Differential Transform Method for Solving gas dynamics Equation. *International Journal of Contemporary Mathematical Sciences*, 5(22), pp. 1091-1096.

Keskin, Y. and Oturanc, G. (2010c). Reduced Differential Transform Method for Generalized KDv Equations. *Mathematical and Computational Applications*, 15(3), pp. 382-393.

Kinnuner, J. (2017). Partial Differential Equations. *Department of Mathematics and Systems Analysis, Aalto University.*

Mohyud-Din, S. T. (2009). Solving Heat and Wave-Like Equations Using He's Polynomials. *Mathematical Problems in Engineering*, 2009(427516), pp. 1-12.

Narasimhan, T. N. (1999). Fourier's Heat Conduction Equation: History, Influence, and Connections. Reviews of Geophysics, 37(1), pp. 151-172.

Noor, M. A. and Mohyud-Din, S. T. (2008). Modified Variational Iteration Method for Heat and Wave-Like Equations. *Acta Applicandae*, 104(3), pp. 257-269.

Wazwaz, A. M. and Gorguis, A. (2004). Exact Solution for Heat-Like and Wave-Like Equations with Variable Coefficients. *Applied Mathematics and Computation*, 149(1), pp. 15-29.