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Elzaki Transform Method for the Fokker-Planck Equation

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ABSTRACT

In this paper, a numerical technique for the solution of Fokker-Planck equation (FPE) has been considered. For this purpose, the Elzaki transform method (ETM) has been adopted for both linear and nonlinear one-dimensional Fokker-Planck equation. The work includes some model equations from which different versions of the FPE were derived. Numerical evidences obtained (using Maple 18 software) showed that the convergence of the method is absolute.

1. INTRODUCTION

A random noise is a random fluctuation experienced in a chaotic or dynamic system (Kloeden and Platen, 1992). Thus, a dynamic system is a stochastic process if it is under the influence of random fluctuations. These fluctuations are generally referred to as stochastic processes and their sudden appearance could implicate the action of degree of freedom (Yamapi et al., 2012). Thus, a stochastic differential equation (SDE) is a non-deterministic differential equation that models a stochastic process whose terms are confined to a random fluctuation, yielding a result that is stochastic (Kushner, 1977; Chang, 1987).

The randomness of a system could be as result of the presence of some chaotic elements or factors of nonlinearities in the system. These could be fluctuations of market dynamics such as exchange rate or ecosystem of aquatic lives. According to an analysis provided by the Money Show (2019), the current fluctuations in the market forces (capital and exchange rate) in Nigerian economy could be blamed on the territorial border closure. The study of these dynamics is necessary in analyzing the system efficiency and in assessing the performances of many state variables. SDE is very useful to researchers as a tool for the analysis of random fluctuations and periodic processes that occur in real-life situations. Numerical applications of their findings help in resolving the system randomness. This is aided by the use of computer application packages for large scale problems. The SDEs have several applications in many areas of science and engineering (Kloeden and Platen, 1992).

There are several forms of FPEs representing the various areas of applications (Plyukhin 2008). Basically, the general form of the FPE is the nonlinear form given as:

$$\frac{\partial}{\partial t}p(x,t) = \sum_{i,j=1}^{R} F_{ij}\left\{\frac{\partial}{\partial x_i}[p(x,t)]^r\right\} - \sum_{i=1}^{R}\frac{\partial}{\partial x_i}[G_i(x,t)p(x,t)] \quad (1)$$

Where $j, i = 1(1)R,\ G_i(x,t)$ is subject to an external force and F_{ij} denotes the diffusion coefficients (which can be spatial and time dependent). with $G_i(x,t) = X_i(x,t),\ \ F_{ij} = \frac{1}{2}\sum_j^d \delta_{ij}(x,t)\delta_{ij}^T(x,t),$ $p(x,t) = \sqrt{2}\,x^{0.5(2\delta-1)}\,\tau(\beta,u)\ \beta = t, u = \frac{2\sqrt{2}}{28+1}x^{0.5}(2\delta+1)$When $r > 2,$ Equation (1) is most relevant in the percolation of gases via porous media and for r = 2, it is applied for the analysis of the saturated regions in the porous media (Stran, 1968; Zhang, 2008).

Consequently, the nonlinear FPE in one variableis given as (Hemeda and Eladdad, 2018):

$$\frac{\partial u}{\partial t} = \left[-\frac{\partial}{\partial x}A_1(x,t,u)u + \frac{\partial^2}{\partial x^2}A_2(x,t,u)u\right] \quad (2)$$

with the prescribed initial conditions as:

$u(x,0) = g(x), x \in \mathbb{R}$

Where $u(x,t)$ is the distribution function, $A_1(x,t,u)$ is the drift coefficient and $A_2(x,t,u)$ is the diffusion coefficient

Gaviraghi et al. (2017) adopted the splitting implicit- explicit method to solve FPE by comparing and integrating two different discretization methods. Risken (1996) also derived some solution schemes for FPE. Arauujo and Filho (2012) also provided a general solution to FPE.

The FPE as SDE has applications in different sciences and in the financial market as it models investment strategies, prices and other market dynamics in the financial system. The idea of applying SDE to investment finance was first suggested by Merton (1973). Merton's idea was a decision between two classes of investment (non-risky and the risky). Merton was of the opinion that an investor will maximize some classes of utility functions such as wealth or cash flow by implementing some investment strategy and minimize possible hiccups such as bankruptcy. His model was formulated from the exponential growth rate equation which was an ordinary differential equation (Equation 3).

$$\frac{dp_s}{dt} = \alpha p_s,\ \alpha > 0, \quad (3)$$

In this case, p_s is the price of safe investment. It was also observed in Merton (1973) that noisy fluctuation in stock exchange market is proportional to the share price. The FPE has been employed to model this situation. Several methods abound for solving the FPE of financial investment with respect to market capitalization, determination of all-share indices in option pricing and market segmentation. Among these methods, numerical methods have been found to offer solution with a rapid converging series in the approximation of the exact solution. Many of these methods are very explicit in application requiring no perturbation, hidden transformation or linearization.

However, according to Mamadu and Njoseh (2017), a major disadvantage of these analytic methods is that they are prone to computational and truncation errors due to the rigorous procedures in their solution processes. Many stochastic models resulting in the FPEs have been solved using the finite element method (FEM). Stilianos (2008) was one of those who used FEM. Apart from using FEM, there are several numerical approaches that have been adopted by other researchers.

This paper aims to use ETM to solve the FPE and investigate the rate of convergence of the method.

2. Elzaki Transform Method

An Elzaki transform of the function g(t) in the set,

$$\Omega=\{f(t): \exists N, a_1 \; and \; a_2 > 0: |f(t)| < Me^{|t|\alpha_j}, if t \in (-1)^n x[0,\infty)\} \tag{4}$$

is defined as (Mamadu and Ojarikre, 2019):

$$E[f(t)] = q\int_0^\infty f(t)e^{-t/q}dt = T(q), q \in (-a_1, a_2) \tag{5}$$

2.1. Properties of Elzaki transforms method

Some important properties for this research are outline as follows:

i. $E[t^n] = n!\, q^{n+2}$
ii. $E^{-1}[q^{n+2}] = \frac{t^n}{n!}$
iii. $T_n(x,q) = \frac{T(x,q)}{q^n} - \sum_{k=0}^{n-1} q^{2-n+k}\frac{\partial^k y(x,0)}{\partial t^k}$, n is the order of the highest derivative

3. Construction of Elzaki Transform Method for FPE

Here, the ETM for resolving the generalized case of the FPE is developed. Applying the ETM on both sides of Equation (2) results in:

$$E\left[\frac{\partial}{\partial t}u(x,t)\right] = E\left[-\frac{\partial}{\partial x}A_1(x,t,u)u_n(x,t) + \frac{\partial^2}{\partial x^2}A_2(x,t,u)u_n(x,t)\right] \tag{6}$$

By property 2.1 (iii), Equation 6 yields:

$$E[u(x,t)] = \sum_{k=0}^{n-1} q^{2-n+k}\frac{\partial^k u(x,0)}{\partial t^k} + rE\left[-\frac{\partial}{\partial x}A_1(x,t,u)u_n(x,t) + \frac{\partial^2}{\partial x^2}A_2(x,t,u)u_n(x,t)\right] \tag{7}$$

Deriving the inverse on both sides of Equation (7), gives:

$$u(x,t) = E^{-1}\left[\sum_{k=0}^{n-1} q^{2-n+k}\frac{\partial^k u(x,0)}{\partial t^k} + rE\left[-\frac{\partial}{\partial x}A_1(x,t,u)u_n(x,t) + \frac{\partial^2}{\partial x^2}A_2(x,t,u)u_n(x,t)\right]\right] \tag{8}$$

However, the approximate solution of the analytic solution for the generalized FPE is given by:

$$u(x,t) = \textstyle\sum_{m=0}^{\infty} u_m(x,t) \tag{9}$$

Approximating Equation (8) yields:

$$\sum_{n=0}^{N} u_n(x,t) = E^{-1}\left[\sum_{k=0}^{n-1} q^{2-n+k}\frac{\partial^k u_n(x,0)}{\partial t^k}\right] + E^{-1}\left[rE\left[-\frac{\partial}{\partial x}A_1(x,t,u)u_n(x,t) + \frac{\partial^2}{\partial x^2}A_2(x,t,u)u_n(x,t)\right]\right] \tag{10}$$

A comparison of both sides of Equation (10), yields the recurrence relations:

$$u_0(x,t) = E^{-1}\left[\textstyle\sum_{k=0}^{n-1} q^{2-n+k}\frac{\partial^k u_n(x,0)}{\partial t^k}\right] \tag{11}$$

$$u_{n+1}(x,t) = E^{-1}\left[qE\left[-\frac{\partial}{\partial x}A_1(x,t,u)u_n(x,t) + \frac{\partial^2}{\partial x^2}A_2(x,t,u)u_n(x,t)\right]\right] \tag{12}$$

Thus, the components u_n for $n \geq 0$ can be computed using Equations (11) and (12).

Hence, the approximate solution of the generalized FPE is given as:

$$u(x,t) = \textstyle\sum_{n=0}^{\infty} u_n(x,t) \tag{13}$$

4. Existence and Uniqueness of Solution to Fokker Planck Equations

Here it is shown that the ETM solution obtained in Equation (13) exists and has a unique solution with the following results.

Theorem 1: Let $0 \leq r \leq 3$, then the FPE of Equation (1) has a unique solution.

Proof: Let $y \text{ and } y^*$ define two separate solutions of Equation (1), then:

$$\left|y - y^*\right| = \left|\sum_{r=0}^{m} a_r x^r - \sum_{k=0}^{m} a_k x^k\right|$$

$$\leq \sum_{r=0}^{m} x^r\left|a_r - a_r^*\right| + \sum_{k=0}^{m}\left|a_k - a_k^*\right| \leq r(l+l)\left|y - y\right|$$

by unifying variables

$$= r\left|y - y^{*}\right|$$

Since $0 \leq r \leq 1, \quad then \quad \left|y - y^{*}\right| = 0 \quad implies \;\; y = y^{*}$

Theorem 2: If the solution $\sum_{n=0}^{\infty} u_n(t)$ is convergent such that $u_n(t)$ is the solution derived through the ETM for Equation (1), then it must be exact solution.

Proof: Convergence of $\sum_{n=0}^{\infty} u_n(t)$ implies $\lim_{n \to \infty} u_n(t) = 0$

Consider the series $\sum_{n=0}^{\infty} u_{n+1}(t)$

$$\sum_{n=0}^{\infty} u_{n+1}(t) = \sum_{n=0}^{\infty} (u_{n+1}(t) - u_n(t)) = \lim_{n\to\infty} u_{n+1}(t) = \begin{cases} 0, n \leq 1 \\ 1, \; n > 1 \end{cases}$$

Similarly, using Equations (5) and (8), yields:

$$\sum_{n=0}^{\infty} u_{n+1}(t) = \sum_{n=0}^{\infty} (A_{n+1}(t) - A_n(t)) = \lim_{n\to\infty} A_{n+1}(t) = \begin{cases} 0, n \leq 1 \\ 1, \; n > 1 \end{cases}$$

5. Numerical Examples

Example 1:

Let$A_1(x,t,u) = -1, A_2(x,t,u) = 1$ and $r(x) = x, x \in \mathbb{R}$, as in Equation (2). The analytic solution will be given as $u(x,t) = x + t$.

Adopting the ETM:

$u_0(x,0) = x$

$$u_{n+1}(x,t) = E^{-1}\left[qE\left[\frac{\partial}{\partial x}u_n(x,t) + \frac{\partial^2}{\partial x^2}u_n(x,t)\right]\right]$$

$k = 0$:

$$u_1(x,t) = E^{-1}\left[qE\left[\frac{\partial}{\partial x}u_0(x,t) + \frac{\partial^2}{\partial x^2}u_0(x,t)\right]\right] = E^{-1}[qE[1]]$$

By implementing the ETM transformation property 2.1 (ii):

$u_1(x,t) = E^{-1}[q^3] \Rightarrow E^{-1}[q^{n+2}] = t$

Similarly, for $k \geq 1, u_{n+1}(x,t) = 0$. Therefore, the computed solution is given as:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = x + t.$$

Example 2:

Let$A_1(x,t,u) = \left(\frac{4u(x,t)}{x} - \frac{x}{3}\right), A_2(x,t,u) = u(x,t)$ and $r(x) = x^2, x \in \mathbb{R}$ in (.2). The analytic solution is given as u$(x,t) = x^2 e^t$.

Using the ETM results in:

$$u(x,0) = x^2$$

$$u_{n+1}(x,t) = E^{-1}\left[rE\left[\frac{\partial}{\partial x}\left(\frac{4u(x,t)}{x} - \frac{x}{3}\right)u_n(x,t) + \frac{\partial^2}{\partial x^2}u_n^2(x,t)\right]\right]$$

Executing the entire process as in Example 1, the following approximations were obtained:

$$u_1(x,t) = x^2 t, u_2(x,t) = x^2\frac{t^2}{2!}, u_3(x,t) = x^2\frac{t^3}{3!}, \ldots$$

Thus:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = x^2 + x^2 t + x^2\frac{t^2}{2!} + x^2\frac{t^3}{3!} + \cdots = x^2 e^t.$$

$x^2 e^t$ is the analytic solution with $u(x,0) = x^2$and applying ETM and comparing the result against exact solution, same solution is obtained.

Table 1: Comparison of results between exact solution and ETM at $t = 0(0.1)$ for example 2

x	$exact\ solution$	ETM	$error$
0.00	0.0000000	0.0000000	0.0000E+00
0.1	0.0100000	0.0100000	0.0000E+00
0.2	0.0400000	0.0400000	0.0000E+00
0.3	0.0900000	0.0900000	0.0000E+00
0.4	0.1600000	0.1600000	0.0000E+00
0.5	0.2500000	0.2500000	0.0000E+00
0.6	0.3600000	0.3600000	0.0000E+00
0.7	0.4900000	0.4900000	0.0000E+00
0.8	0.6400000	0.6400000	0.0000E+00
0.9	0.8100000	0.8100000	0.0000E+00
1.00	1.0000000	0.0000000	0.0000E+00

The results obtained from implementing ETM for the analytic solution of nonlinear FPEare displayed in Table 1. From Table, the rate of convergence of the method for the solution of the FPE is absolute in a few iterations.

6. CONCLUSION

The FPE is an important mathematical model in various applications whose solution is always difficult to arrive at analytically. Thus, numerical methods have given researchers the means of finding the approximate solution to this model. Therefore, the results of this work have shown that the ETM is a good numerical technique for solution of FPK. The method also exhibits an absolute convergence. This implies that the ETM in this concept can be termed an analytical method, and also a numerical iterative scheme for the solution of FPE.

5. CONFLICT OF INTEREST

There is no conflict of interest associated with this work.

REFERENCES

Arauujo, M. T. and Filho, E. D. (2012). A General Solution of Fokker-Planck Equation. *Journal of Statistical Physics*, 146, pp. 610-619.

Chang, C.C. (1987). Numerical Solution of Stochastic Differential Equations with Constant Diffusion Coefficients. *Mathematical Computation,* 49, pp. 523-542.

Gaviraghi, B., Annunziato, M. and Borzì, A. (2017). Analysis of splitting methods for solving a partial integro-differential Fokker–Planck equation. *Applied Mathematics and Computation*, *294*, 1-17.

Hemeda, A.A. and Aladdad, E.E. (2018). New Iterative Methods for solving Fokker-Planck Equation. *Hindawi Journals: Mathematical Problems in Engineering*, 2018.

Kloeden P.E. and Platen, E. (1992). *Numerical solution of stochastic differential equations.* Springer-Verlag.

Kushner, H.J. (1977). *Probability Methods for Approximation in Stochastic Control and for Elliptic Equations*. Academic Press, New York.

Mamadu, E.J. and Njoseh, I.N. (2017). Solving delay differential equations by Elzaki transform method. *Boson Journal of Modern Physics,* 3(1), pp. 214-218.

Mamadu, E.J. and Ojarikre, H.I. (2019). Reconstructed Elzaki transform method for delay differential equations with Mamadu-Njoseh Polynomials. *Journals of Mathematics and System Science*, 9, pp. 41-45.

Merton, R.C. (1973). Theory of rational option pricing. *Bell. Journal of. Economics and. Management. Science*, 4, pp. 141-183**.**

Money Show (2019). AIT current Affairs Program, 29th October, 2019.http://m.youtube.com>watch

Plyukhin, A. V. (2008). Generalized fokker-planck equation, brownian motion, and ergodicity. *Physical Review E*, *77*(6), 061136.

Risken, R. (1996). *The Fokker-Planck Equation: Methods and Applications*. Springer, New York

Stilianos, M. (2008). Standard Galerkin Formulation with high order Lagrange Finite Elements for Option Pricing. *Appliied Mathematics and Computations,* 195(2), pp. 707-720.

Stran, G. (1968). On the Construction and Comparison of different Schemes. *SIAM Journal of Numerical Analysis*, 5, pp. 506-517.

Yamapi, R., Filatrella, G., Aziz-Alaoui, M.A. and Cerdeira, H.A. (2012). Effective Fokker-Planck Equation for Birhythmic Modified van der Pol Oscillator. *Nonlinear Science; Chaotic Dynamics: Chaos*, 22, p. 043114.

Zhang, W. (2008). Analytical Solution of General one-dimensional Fokker-Planck Equation. *International Journal of Control,* 49(6), pp. 2085-2091.